Effects of hyperbolic rotation in Minkowski space on the modeling of plasma accelerators in a Lorentz boosted frame

J.-L. Vay,[1,a)] C. G. R. Geddes,[1] E. Cormier-Michel,[2] and D. P. Grote[3]
[1]*Lawrence Berkeley National Laboratory, Berkeley, California 94720, USA*
[2]*Tech-X Corporation, Boulder, Colorado 80303, USA*
[3]*Lawrence Livermore National Laboratory, Berkeley, California 94550, USA*

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The effects of hyperbolic rotation in Minkowski space resulting from the use of Lorentz boosted frames of calculation on laser propagation in plasmas are analyzed. Selection of a boost frame at the laser group velocity is shown to alter the laser spectrum, allowing the use of higher boost velocities. The technique is applied to simulations of laser driven plasma wakefield accelerators, which promise much smaller machines and whose development requires detailed simulations that challenge or exceed current capabilities. Speedups approaching the theoretical optima are demonstrated, producing the first direct simulations of stages up to 1 TeV. This is made possible by a million times speedup thanks to a frame boost with a relativistic factor γ_b as high as 1300, taking advantage of the rotation to mitigate an instability that limited previous work. © *2011 American Institute of Physics*. [doi:10.1063/1.3559483]

Laser driven plasma waves produce accelerating gradients orders of magnitude greater than standard accelerating structures (which are limited by electrical breakdown).[1,2] High quality electron beams of energy up to 1 GeV have been produced in just a few centimeters[3–6] with 10 GeV stages being planned as modules of a conceptual future high energy collider,[7] and detailed simulations are required to realize the promise of much shorter particle accelerators using this technique.[8] Such simulations challenge or exceed current capabilities, in particular for high energy stages at GeV energies and beyond.

Theory predicts that for the intense lasers ($a \gtrsim 1$) typically used to achieve high gradients, the laser depletes its energy over approximately the same length $L_d=\lambda_p^3/2\lambda_0^2$ over which the particles dephase from the wake, where $\lambda_p=\sqrt{\pi/r_e n_e}$ and λ_0 are, respectively, the plasma and vacuum laser wavelengths, and r_e and n_e are, respectively, the electron classical radius and density in the plasma.[1,2] As a result of beam dephasing and laser depletion, the maximum bunch energy gain scales approximately as λ_p^2 and $1/n_e$, which implies that higher energy stages operate with longer plasmas, rendering computer simulations more challenging as the ratio of longest to shortest spatial lengths of interest L_d/λ_0 rises. In fact, direct explicit three dimensional (3D) simulations of 10 GeV stages, which will operate in m-scale plasmas at $\sim 10^{17}$/cc densities, were considered until recently beyond the current state of the art.[8,9]

The Particle-In-Cell (PIC) technique with a boosted frame[8,10–20] accurately resolves the wavelength shifting and broadening that occurs as the laser depletes, offering advantages over other models (e.g., envelope, quasistatic) while providing the speed required for direct simulation of 10 GeV and beyond laser plasma wakefield accelerators (LPAs) to accurately model laser and beam transverse oscillations.[21] It has been predicted that first principles PIC modeling of LPAs can be sped up by 4–6 orders of magnitude for stages in the energy range of 10 GeV–1 TeV, respectively.[19,20] Practical limitations have prevented reaching these speedups, including a violent high-frequency numerical instability, limiting the Lorentz boost to $\gamma_b \leq 100$ and hence the speedup to three orders of magnitude.[8,13,16,19,20] Understanding the physics of the Lorentz boost effects is important to understand laser and plasma wave dynamics and to accurately simulate such accelerators.

In this letter, we report an analysis of how the hyperbolic rotation of the laser propagation in the plasma column in space-time modifies its physical manifestation and spectral content. We elucidate why for particular values of the boost velocity this physical modification permits numerical techniques such as smoothing to be used aggressively, enabling modeling at the maximum attainable speedup and accuracy and explaining previous observations.[19,20] The technique is applied to simulation of LPAs (assuming in all calculations a laser wavelength $\lambda=0.8$ μm, normalized vector potential $a_0=1$, and other parameters scaling with the plasma density[9,22]), demonstrating direct explicit simulations of stages from 0.1 GeV up to 1 TeV for the first time, using a Lorentz boosted calculation frame with γ_b as high as 1300. This verifies the performance and energy gain scaling[9,22] of plasma accelerator stages with deep laser depletion into the 1 TeV range and provides the tools for detailed designs for upcoming 10 GeV experiments such as BErkeley Lab Laser Accelerator (BELLA).[23]

Effect of the hyperbolic rotation in Minkowski space. The effects of the Lorentz transformation on the laser and wake propagation is illustrated by considering a 0.1 GeV LPA stage[9,22] taken from simulations using the PIC code Warp.[24] The spatial effects are demonstrated by snapshots of the fields in the laboratory and boosted calculation frames.

a)Electronic mail: jlvay@lbl.gov.

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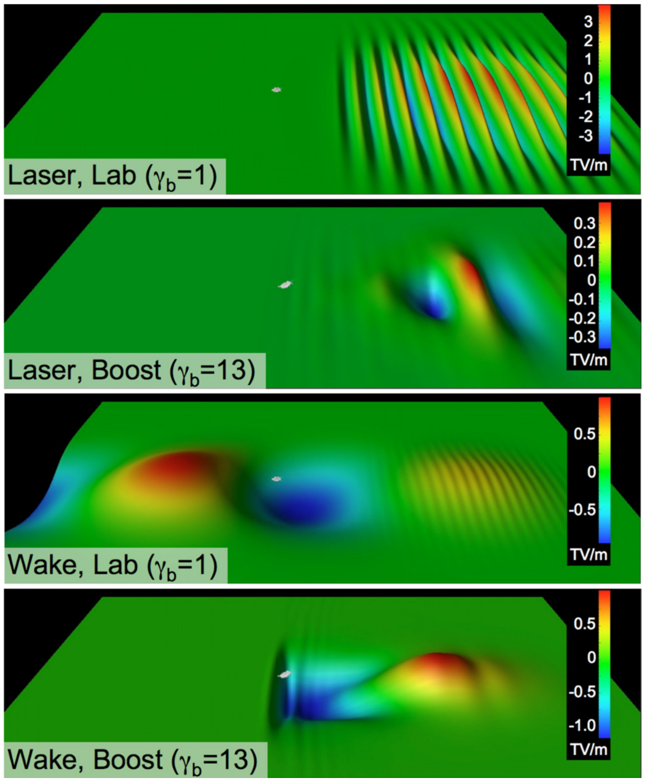

FIG. 1. (Color online) Colored surface rendering of the transverse (laser) and longitudinal (wake) electric fields from a 2–1/2D Warp simulation of a 0.1 GeV LPA stage in the laboratory frame (left) and a boosted frame at $\gamma_b = 13 \approx \gamma_{wake}$ (right), with the beam (white) in its early phase of acceleration. The laser and the beam are propagating from left to right.

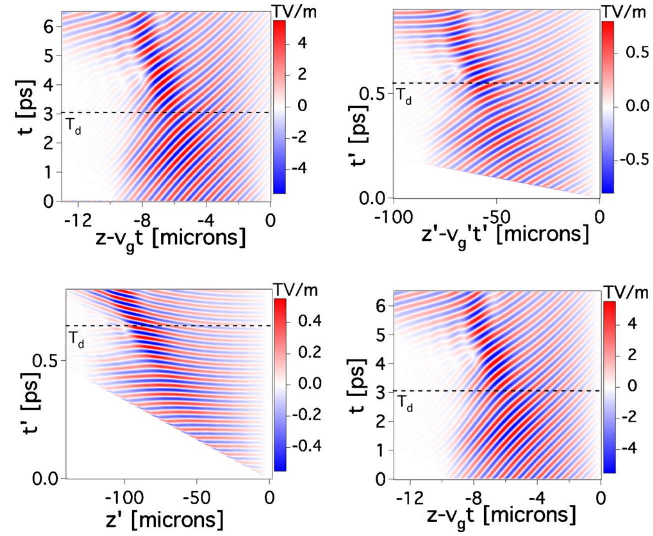

FIG. 2. (Color online) Color plots of the laser field history on axis in a moving window propagating at the laser group velocity from simulations of a 0.1 GeV LPA: (top-left) in the laboratory frame; (top-right) in a boosted frame at $\gamma_b = 5$; (bottom-left) in a boosted frame at $\gamma_b = 13$; and (bottom-right) in a boosted frame at $\gamma_b = 13$ plotted in laboratory coordinates after Lorentz transformation of the data.

The space-time correlations and spectral effects are shown by histories of the field evolution.

Figure 1 shows surface renderings of the transverse and longitudinal electric fields, respectively, as the beam enters its early stage of acceleration by the plasma wake, from calculations in the laboratory frame and a Lorentz boosted frame with $\gamma_b = 13$ (approximately the γ of the laser group velocity in the plasma $\gamma_g \approx 13.2$). The two snapshots offer strikingly different views of the same physical processes: in the laboratory frame, the wake is fully formed before the beam undergoes any significant acceleration, and the laser (Fig. 1, top-left) is easily recognizable (i.e., its shape is only slightly distorted by the plasma) and leaves a visible imprint on the wake (longitudinal) field (Fig. 1, bottom-left); in the boosted frame, the beam is accelerated as the plasma wake develops, the laser (Fig. 1, top-right) is not easily recognizable (i.e., its shape is highly distorted by the plasma) and no evident imprint is left on the wake field (Fig. 1, bottom-right).

The Lorentz transformation can be described as a hyperbolic rotation in Minkowski space and its rotational effect is explicitly visible in Fig. 2, where the physics underlying the differences between $\gamma_b = 1$ and $\gamma_b = 13$ views of the wake is illustrated by histories of the (transverse) laser field on the longitudinal axis, from simulations performed using the laboratory frame and boosted frames at $\gamma_b = 5$ and 13 as the laser propagates through the plasma past the laser depletion point T_d. Simulations with $\gamma_b = 1$ and 5 used a moving window propagating at the group velocity v_g of the laser in the plasma and the data are plotted in the frame of the (Galilean) moving window. The simulation with $\gamma_b = 13$ did not use a moving window as it was performed very near the group velocity of the laser in the plasma $\gamma_g \approx 13.2$. The data from the boosted frame simulation at $\gamma_b = 13$ is presented in the boosted frame as well as in the laboratory frame moving window (after Lorentz transformation), allowing direct comparison with the simulation calculating in the laboratory frame. The calculation with the boosted frame at $\gamma_b = 13$ was ~200 times faster than the calculation with the laboratory frame, as expected.[10] The agreement between the two is nonetheless excellent (comparing top-left plot to bottom-right plot in Fig. 2), confirming the accuracy of the calculation in the boosted frame. The group velocity of the laser in the plasma is always below the speed of light in vacuum while the laser phase velocity is always above it, resulting in oblique stripes in the laboratory frame plot. As γ_b rises, the stripes rotate according to the rules of the Lorentz transformation, eventually becoming nearly perpendicular to the time axis as γ_b nears γ_g (bottom-left plot in Fig. 2), at which point the laser group velocity approaches zero and the phase velocity approaches infinity. In effect, the laser oscillations that appear in the laboratory as spatial oscillations propagating in the plasma are transformed into time beating of the field for calculations in frames whose $\gamma_b \approx \gamma_g$. As discussed below, this effect has important consequences for the modeling of full scale stages at 10 GeV or above, and in explaining the successful simulations at high γ_b.[19,20]

Spectral content and mitigation of numerical instability. The spectral content history of the laser field on axis as it propagates through the plasma column is given in Fig. 3 for 0.1 and 10 GeV stages, in the laboratory frame and in the laser group velocity frames ($\gamma_b = 13$ and 130, respectively). In the laboratory frame, the spectral content of the laser is

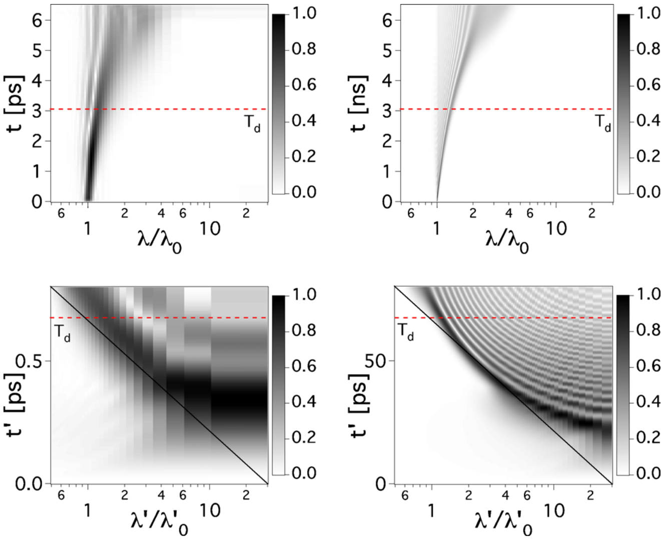

FIG. 3. (Color online) Spectral content history of the laser field on axis as it propagates through the plasma column for 0.1 GeV (left) and 10 GeV (right) stages, in the laboratory frame (top) and near the laser group velocity frames at $\gamma_b = 13$ (bottom-left) and 130 (bottom-right). The length scale (horizontal axis) is normalized relative to the vacuum laser wavelength as given in each respective frame.

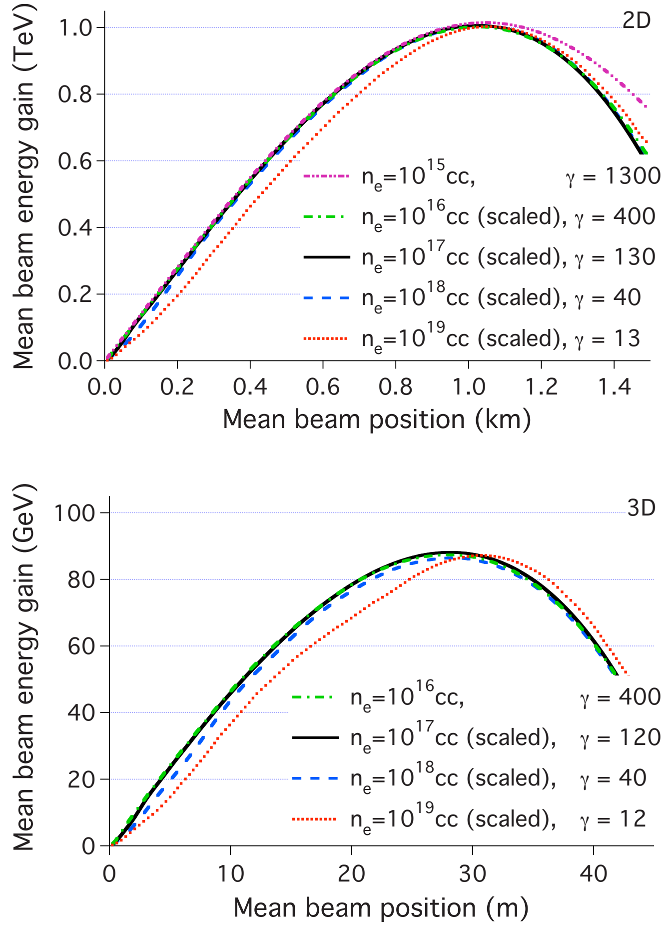

FIG. 4. (Color online) Verification of the scaling (Refs. 9 and 22) of electron beam energy gain vs longitudinal position (in the laboratory frame) from direct simulations at $n_e = 10^{19}$ cc down to 10^{15} cc (energy gains from 0.1 to 1 TeV), using Lorentz boosted frames of reference at γ_b between 12 and 1300, in 2–1/2D (top) and 3D (bottom). Energies and lengths from the lower energy stages were scaled to the highest energy stage to allow for direct comparison.

concentrated initially in a narrow band around the laboratory frame vacuum laser wavelength λ_0, then spreads due to dispersion and depletion effects.[2] In the frame of the laser group velocity, the spectral content is initially localized at wavelengths of several wake frame vacuum laser wavelength λ'_0, then progressively fills shorter-wavelength spectral regions over propagation. Hence, at higher boost, the spectral content the simulation must resolve is confined to longer wavelengths, especially early in laser propagation.

The shifting of spectral content to long wavelength due to rotation allows us to control a violent high-frequency numerical instability that limited performance in previous work, as reported by various authors[15,16,25] for boosts at $\gamma \gtrsim 100$ in two dimensions (2D) and $\gamma \gtrsim 50$ in 3D. The instability develops at the front of the plasma column introducing unphysical noise, in particular in the range of $\sim 1-20$ times the grid period, which can grow to sufficient amplitude to overwhelm the laser pulse. Extensive tests were performed with Warp to investigate simulations of downscaled 0.1 GeV and full scale 10 GeV LPA stages.[20] The presence and growth rate of the instability was observed to be sensitive to the resolution (slower growth rate at higher resolution), to damping of high frequencies, to smoothing of short wavelengths, and to the boost value (stronger instability at higher boost). A key finding was that the frame at $\gamma_b = \gamma_g$ allows the highest levels of filtering and damping and that as a result, simulations at this boost velocity accurately reproduced the laboratory frame result while realizing the theoretically predicted[10] speedups. Simulations at other values of γ_b failed unless γ_b (and hence speedup) was kept relatively small (as in previous work[8,11–18]) because filtering sufficient to control the instability affected the laser pulse and hence electron beam energy.[20] This is a direct consequence, and benefit, of the hyperbolic rotation effect from the Lorentz boost seen in Figs. 2 and 3, as in this frame the spectral content of the laser is confined to the longest wavelength region, allowing suppression of the unphysical short-wavelength instability without affecting laser propagation. Other numerical limits that have restricted the boost performance in past simulations (laser initialization, statistics) are discussed elsewhere.[19]

Modeling of up to 1 TeV stages. The observed optimal boost frame was used to directly simulate stages with gain energy as high as 1 TeV in 2–1/2D and 100 GeV in 3D, using a Lorentz boosted frame with γ_b as high as 1300 (see Fig. 4), offering for the first time direct verification of the scaling of LPAs into the 1 TeV range for deeply depleted stages. The 2D simulations closely reproduced the performance predicted by density scaling.[22] The fast turnaround time of the boost simulations was used to adjust the phase of the beam behind the laser in 3D, compensating for mildly stronger depletion,[9] and performance again scaled as anticipated, yielding 90% of the 2D energy. The full scale 100 GeV class 3D run used $\gamma_b = 400$ allowing a speedup over 100 000-fold, exploiting for the first time the full theoretical potential of the boosted frame technique. Assuming the use of a few thousands of CPUs, a simulation that would have required an impractical several decades to complete using the laboratory

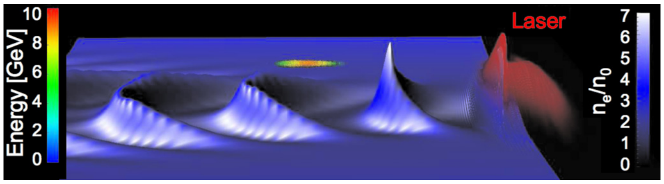

FIG. 5. (Color online) Snapshot from a 10 GeV LPA stage boosted frame simulation. The image shows an externally injected electron bunch (middle) riding a density wake excited by an intense laser pulse (right), propagating in a 0.65 m long plasma channel.

frame was completed in only 4 h using 2016 CPUs of the Cray system at National Energy Research Scientific Computing Center (NERSC). The speedup of the 2–1/2D 1 TeV stage simulation is over a million.

Direct simulations were conducted for the first time of deeply depleted 10 GeV beam loaded stages at full scale, providing detailed design of experiments on new lasers such as BELLA[22,23] as well as next generation LPA stages and collider modules.[7] Figure 5 shows the density wake and externally injected electron beam accelerated in a stage where a laser pulse with a duration of 67 fs was focused to a Gaussian transverse spot size $w_0=89$ μm at the entrance of a plasma channel. The channel had a parabolic transverse density profile with 60% of the linear matched amplitude,[2] and was 0.65 m long. The on-axis density was tapered: $n(x)=n_0(1.32x+1)$ with $n_0=10^{17}$ cm^{-3} to compensate dephasing for the bunch loaded in the second accelerating bucket and hence increase energy gain.[22] Electrons were externally injected with an initial energy of 100 MeV and an initial emittance of 63 mm mrad. The large input emittance was chosen to set the beam radius for efficient beam loading and for emittance matching to the wake focusing fields.[22] At the exit of the structure, electrons with energy of up to 9.2 GeV were observed. The depression in the density wake is due to self-consistent beam loading by the injected electron bunch. The time projected energy spread and normalized emittance when exiting the plasma channel were 15% and 57 mm mrad, respectively. The slice energy spread and emittance of a slice at 9 GeV were 1% and 54 mm mrad. Whereas these values are larger than acceptable for collider and light source applications, it has been shown that lower emittance bunches can be accelerated by using high order laser modes to control the transverse focusing forces[26] and lower energy spread by controlling beam loading.[22] Future work will aim at optimizing the phase space properties of the bunch, including optimization of taper and use of higher order laser modes to minimize emittance.

In summary, it is shown that the physical effects of hyperbolic rotation in Minkowski space on laser propagation in a plasma column alter the manifestation of the laser spectrum. This allows boosted frame simulations to use very high boost by enabling high levels filtering, suppressing instabilities while maintaining accuracy. Efficiently depleted stages at 10 GeV were simulated using this technique, providing designs for next generation laser facilities, and direct simulations of stages in the range of 0.1 GeV–1 TeV have been performed. This verifies for the first time the performance of plasma accelerators into the 1 TeV range for deeply depleted stages and shows that the Lorentz boosted frame technique is applicable to the 10 GeV stages and beyond at full efficacy.

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